End-Semestral Exam
Algebra-IV
B. Math - Second year
2014-2015

Time: 3 hrs
Max score: 100

Answer all questions.

(1) (i) Prove that there exists no field extension F of $\mathbb{R}$ such that $[F:\mathbb{R}]$ is odd.
(ii) Give an example of a field of order 16 which is a quotient of $\mathbb{Z}_2[x]$. (5+5)

(2) Let p be a prime number and $a, b \geq 1$. Prove that $\mathbb{F}_{p^a}$ can be embedded into $\mathbb{F}_{p^b}$ (that is, is isomorphic to a subfield of $\mathbb{F}_{p^b}$) if and only if a/b. (10)

(3) Suppose $K|F$ is a Galois extension and $F'|F$ is any extension. Then show that the compositum KF' of fields K and F' is a Galois extension over F', with Galois group isomorphic to a subgroup of $Gal(K|F)$. (10)

(4) Let $E|\mathbb{Q}$ be a finite extension which contains no proper abelian extension of $\mathbb{Q}$. Show that, if ζ_n is a primitive nth root of unity, then $Gal(E(\zeta_n)|E) \cong (\mathbb{Z}/(n))^\times$ for all $n \geq 1$, or equivalently $[E(\zeta_n) : E] = \phi(n)$, where ϕ denotes the Euler's phi function. (10)

(5) (i) Let $K|F$ be a finite extension. Show that K is a simple extension of F if and only if there exist only finitely many subfields of K containing F.
(ii) Hence deduce that if $K|F$ is finite and separable, then $K|F$ is simple. (10+5)

(6) (a) Define discriminant D_f of a polynomial $f(x)$.
(b) Let $char(F) \neq 2$ and $f(x) \in F[x]$ be a separable cubic with discriminant D_f. Show that if α is one root of $f(x)$, then a splitting field of $f(x)$ over F is $F(\alpha, \sqrt{D_f})$.
(c) Deduce that if $f(x)$ is a reducible cubic then its splitting field over F is $F(\sqrt{D_f})$.
(d) Give a counter example to show that the result does not hold over a field of characteristic 2. (3+8+2+5)

(7) Prove that a regular n-gon can be constructed by straightedge and compass if and only if $\phi(n)$ is a power of 2. (12)

(8) Define finite solvable groups.
Show that the polynomial $f(x)$ can be solved by radicals if and only if its Galois group is a solvable group. (3+12)